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NUMERICAL SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY ITERATIVE METHODS

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ABSTRACT

Numerical method is the important aspects in solving real world problems that are related to mathematics, science, medicine, business are very few examples. Numerical method is the area related to mathematics and computer science which create, analysis and implements algorithm to numerically solve the system of linear equations. Numerical methods commonly involve an iterative method (as to find roots). They are now mostly used as preconditions for the popular iterative solvers. While it is difficult task solve as it takes a lot of time but it is an interesting part of Mathematics. In this paper the main emphasis on the beginners that how to iterate the solution of numerical to get appropriate results.

KEYWORDS: Jacobi, Gauss-Seidel method, linear equations.

INTRODUCTION

There are several applications in science and engineering where relevant physical law(s) immediately produces a set of linear equations. A system of linear equation (or linear system) is a collection of linear equations involving the same set of variables. A general system of *m* linear equations with *n* unknowns may be written as:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

.

.

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

Numerical method is the branch of mathematics that deals with the appropriate way of finding solution to various mathematical problems. In this age of computer, numerical methods have found a significant role in solving various engineering and scientific problems. Numerical analysis is the study of algorithms that use numerical approximation for the problems of mathematical analysis The system of linear equations has applications in various areas of science such as operational research, physics, statistics, engineering, and social sciences. Equations of this type are necessary to solve for the involved parameters. Iterative methods employ a starting solution and converge to a value that is close to the exact solution by iteratively refining the starting solution. However, iterative methods do not guarantee a solution for all systems of linear equations and they are more frequently used for solving large sparse systems of linear equations.

We started with an approximation to the true solution and by applying the method repeatedly we get better and better approximation till solutions is achieved.

There are two iterative methods for the solving simultaneous equations.
1. Jacobi method
2. Gauss-Seidel method

MATERIALS AND METHODS

1). Jacobi method

The method is illustrated by taking an example

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\}.....(1)$$

After division by suitable constants transposition the equation can be

$$\left.\begin{aligned} x = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}y - \frac{a_{13}}{a_{11}}z \\ y = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x - \frac{a_{23}}{a_{22}}z \\ z = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x - \frac{a_{32}}{a_{33}}y \end{aligned}\right\}......(2)$$

Let us assume $x = 0, y = 0, z = 0$ as first approximation, substituting the values of x, y, z on the right hand side of (2), we get $x = \frac{b_1}{a_{11}}, y = \frac{b_2}{a_{22}}, z = \frac{b_3}{a_{33}}$ this is the second approximation to the solution of the equations again substituting these values of x, y, z in (2) we get a third approximation. The process is repeated till two successive approximations are equal or nearly equal.

Note: Condition for using the iterative method is that the coefficients in the leading diagonal are large compared to the other if these are not so, the interchanging the equations we can make the leading diagonal dominant diagonal.

Example 1: Solve by Jacobi method

$$\left.\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 = 3 \\ -2x_1 + 10x_2 - x_3 - x_4 = 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 = 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 = -9 \end{aligned}\right\}......(3)$$

Solution: Given equations

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4....(3.1)$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4....(3.2)$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4....(3.3)$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3....(3.4)$$

Taking Initial approximation $x_1 = x_2 = x_3 = x_4 = 0$ then

On substituting $x_2 = x_3 = x_4 = 0$ in equation (3.1), we get

$$x_1 = 0.3$$

On substituting $x_1 = x_3 = x_4 = 0$ in equation (3.2), we get

$$x_2 = 1.5$$

On substituting $x_1 = x_2 = x_4 = 0$ in equation (3.3), we get

$$x_3 = 2.7$$

On substituting $x_1 = x_2 = x_3 = 0$ in equation (3.4), we get

$$x_4 = -0.9$$

I. First approximation $x_1 = 0.3$, $x_2 = 1.5$, $x_3 = 2.7$, $x_4 = -0.9$ then

On substituting $x_2 = 1.5$, $x_3 = 2.7$, $x_4 = -0.9$ in equation (3.1), we get

$x_1 = 0.78$

On substituting $x_1 = 0.3$, $x_3 = 2.7$, $x_4 = -0.9$ in equation (3.2), we get

$x_2 = 1.74$

On substituting $x_1 = 0.3$, $x_2 = 1.5$, $x_4 = -0.9$ in equation (3.3), we get

$x_3 = 2.7$

On substituting $x_1 = 0.3$, $x_2 = 1.5$, $x_3 = 2.7$ in equation (3.4), we get

$x_4 = -0.18$

Repeating same above steps for second, third, fourth approximation, and so on.

The result we get are shown in the table below:

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	−0.9
2	0.78	1.74	2.7	−0.18
	0.9	1.908	2.91s6	−0.108
4	0.9624	1.9608	2.9592	−0.036
5	0.9845	1.9848	2.9851	−0.0158
6	0.9939	1.9938	2.9938	−0.006
7	0.9975	1.9975	2.9976	−0.0025
8	0.9990	1.9990	2.9990	−0.0010
9	0.9996	1.9996	2.9996	−0.0004
10	0.9998	1.9998	2.9998	−0.0002
11	0.9999	1.9999	2.9999	−0.0001
12	1.0	2.0	3.0	0.0
13	1.0	2.0	3.0	0.0

Then actual values are x_1=1.0, x_2 =2.0, x_3=3.0, $x_4 = 0.0$

2). Gauss – Seidel method

Gauss-Seidel method, the same set of values of equation (3) is used to verify the result. Here we illustrate that while using this method we have to do less calculation and utilize minimum time to iterate the problem to get appropriate result as compare to Jacobi method. We know say that Gauss-Seidel method is a modification of Jacobi method.

The method is illustrated by taking an example

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \ldots.(4)$$

After division by suitable constants and transposition, the equation can be written as

$$\left.\begin{aligned} x &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} y - \frac{a_{13}}{a_{11}} z \\ y &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x - \frac{a_{23}}{a_{22}} z \\ z &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x - \frac{a_{32}}{a_{33}} y \end{aligned}\right\} \ldots.(4.1)$$

Step1. First we put $y = z = 0$ in first equations (4.1) and $x = c_1$. Then in second equations we put of this values of x *i.e* c_1 and $z = 0$, and obtain y, in the third equation (4.1). We use the values of x and y obtain earlier to get z.

Step2. We repeat the above procedure. In first equations we put values of y and z obtain in step 1, and predetermine x. By using the new values of x and value of z obtained

In step 1 we predetermine y and so on.

In other word, the latest values of the unknowns are used in each step.

Consider the following equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

The above equations can be rewritten as

$$x = \frac{1}{a_1}[d_1 - b_1 y - c_1 z]$$

$$y = \frac{1}{b_2}[d_2 - a_2 x - c_2 z]$$

$$z = \frac{1}{c_3}[d_3 - a_3 x - b_3 y]$$

Initial approximations

$x = x_0, y = y_{0,} z = z_0$

To find $x = x_1$

$$x_1 = \frac{1}{a_1}[d_1 - b_1 y_0 - c_1 z_0]$$

To find $y = y_1$, put $x = x_1$, $z = z_0$

$$y_1 = \frac{1}{b_1}[d_1 - a_2 x_1 - c_2 z_0]$$

To find

$z = z_1$, put $x = x_1, y = y_1$

$$z_1 = \frac{1}{c_1}[d_3 - a_3 x_1 - b_3 y_1]$$ and so on.

Note: 1. The convergence of Gauss-Seidel method is twice as fast as in Jacobi method.

2. If the absolute value of largest coefficient is greater than the sum of the absolute value of the entire remaining coefficient than the method converses for any initial approximation.

Example2: Solve equations (3) by Gauss-Seidel method

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$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Solution: Given equations

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

Taking Initial approximation $x_1 = x_2 = x_3 = x_4 = 0$ then

The result are given in the table.

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9982	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	1.9997	3.0	0.0
7	1.0	2.0	3.0	0.0
8	1.0	2.0	3.0	0.0

Then actual values are x_1=1.0, x_2 =2.0, x_3=3.0, $x_4 = 0.0$

CONCLUSIONS

Jacobi and Gauss-Seidel methods are used for numerical solution of system of linear equation. A numerical analysis start from an initial point and taken a short step forward in time to time find the next solution point. The process continues with subsequent step to map out the solution. Gauss-Seidel method and Jacobi method are iterative methods. Iterative methods are discussed to solve linear systems of equations. It is investigated that Gauss-Seidel method is the most efficient method to solve linear systems of equations. It requires less computational time to converge as compared to other iterative methods. Thus, the Gauss-Seidel method could be considered the more efficient one.

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